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Effect of an alternating electric field on transition radiation

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Abstract. A uniformly moving charged particle that traverses an interface between two media having different dielectric properties loses energy in the form of transition radiation. The present work considers the problem of transition radiation in the presence of an alternating electric field. It is assumed that the particle travels at a uniform relativistic velocity and that the alternating electric field is applied at the boundary between two media in a direction parallel to the velocity of the particle. The method of Fourier transforms is used to solve the modified Maxwell equations with appropriate boundary conditions. Assuming that modulation energy is small compared with mean particle energy and using a linear approximation, the relativistic formula specifying the intensity of transition radiation in the presence of the field is presented in a compact form. It shows that in the presence of the field, the backward transition radiation increases, but forward transition radiation decreases compared with the radiation without a field. The contribution to the radiation due to the field varies linearly with the field strength, but varies inversely with the field frequency.

1. Introduction

A uniformly moving charged particle loses energy in the form of transition radiation when it crosses the boundary between two media. The phenomenon of transition radiation was predicted long ago by Frank and Ginzburg (1945). Its theory has been developed by Garibyan (1958, 1960), Pafomov (1959), Zhelnov (1961), Yakovenko (1962) and many others. Recently, Ginzburg and Tsytovich (1979) have presented a comprehensive review article on the subject. But it appears that the problem, initiated by Diasamidze and Tsikarishvili (1973, to be referred to as DT), namely that of finding the effects of the alternating electric field on transition radiation, has not been completely worked out by any worker so far. DT have inferred that the intensity of transition radiation changes markedly due to the application of the field. Their conclusion regarding a slight reduction in transition radiation due to the weak high-frequency field, a significant reduction in the strong field and a quadratic increase due to the weak low-fequency field is rather doubtful. Moreover, their treatment is non-relativistic. Actually, transition radiation increases with particle energy (Garibyan 1961, Barsukov 1960), and it becomes useful from the application point of view only at relativistic and ultra-relativistic velocities (Zrelov and Ruzicka 1978). So the relativistic generalisation of the problem is extremely important.

In view of the above, we have undertaken a study of the effects of an alternating electric field on transition radiation due to a charge in relativistic motion.

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2. Background

Consider a charged particle with charge e moving with a uniform linear relativistic velocity $\boldsymbol{u}_0$ perpendicular to the boundary separating two media that are characterised by dielectric constants and magnetic permeabilities ε_1 and μ_1 (lower half space) and ε_2 and μ_2 (upper half space) respectively. Assume that the particle crosses the boundary, i.e. the xy plane, at time $t=0$ and that an electric field $\boldsymbol{E}=\boldsymbol{E}_0 \sin \omega_0 t$ is applied in the z direction. Due to the field the velocity of the charge $\boldsymbol{u}_0$ changes to $\boldsymbol{v}(t)$. Using the Lorentz transformation and the formula for relativistic addition of two velocities, we can write the effective velocity of the particle in the field as

$$v_x = 0 = v_y,$$

$$v_z(t) = \frac{u_0 - u' \cos \omega_0 t}{1-(u_0 u'/c^2) \cos \omega_0 t} = \frac{u_0(1-\alpha \cos \omega_0 t)}{(1-\alpha\beta^2 \cos \omega_0 t)}, \tag{1}$$

where

$$\boldsymbol{u}' = \frac{e\boldsymbol{E}_0}{m\omega_0} = (u')_z, \qquad m = \frac{m_0}{(1-\beta_0^2)^{1/2}} = \gamma m_0, \qquad \beta_0 = \left|\frac{\boldsymbol{u}_0}{c}\right|, \qquad \alpha = \frac{u'}{u_0}.$$

Consequently, the charge and current densities in the Maxwell equations become

$$\rho(t) = e\delta(x-h)\delta(y)\delta(z-z(t)), \tag{2}$$

$$\boldsymbol{j}(t) = \rho(t)\boldsymbol{v}(t) = j_z(t), \tag{3}$$

where

$$z(t) = \int v_z(t')\,\mathrm{d}t'.$$

Here we assume that the modulation energy is small compared with the mean particle energy, i.e. $u' \ll u_0$, and so we consider the terms linear in u' only (Risbud and Takwale 1979). Thus, we can write

$$v_z(t) \simeq u_0 - (u/\gamma^3) \cos \omega_0 t \tag{1'}$$

where

$$u = eE_0/m_0\omega_0.$$

Using equation (1′) we obtain

$$z(t) = u_0 t - (u/\gamma^3 \omega_0) \sin \omega_0 t = u_0 t - a \sin \omega_0 t \tag{4}$$

where

$$a = u/\gamma^3 \omega_0.$$

The electromagnetic fields induced by the charge in a medium are given by Maxwell's equations. In terms of the scalar (ϕ) and vector ($\boldsymbol{A}$) potentials, they reduce to

$$\nabla^2 \phi - n^2 \partial^2 \phi/\partial t^2 = -\rho/\varepsilon, \qquad \nabla^2 \boldsymbol{A} - n^2 \partial^2 \boldsymbol{A}/\partial t^2 = -\mu \boldsymbol{j}, \tag{5}$$

where $n = (\varepsilon\mu)^{1/2}$ is the refractive index of the medium and the charge and current densities $\rho(t)$ and $\boldsymbol{j}(t)$ are given by equations (2) and (3) respectively.

Here the problem is tackled by the method of Fourier transforms. Therefore, all quantities such as electromagnetic fields, charge and current densities and potentials, etc, are expressed in the following form:

$$F(r, t) = \int_{-\infty}^{+\infty} F(k, \omega) \exp[\mathrm{i}(k \cdot r - \omega t)]\, \mathrm{d}k\, \mathrm{d}\omega \tag{6}$$

where $F(r, t)$ may be any one of the functions mentioned above. In order to obtain the general solutions, the fields obtained from equation (5) need to be supplemented by the solutions of the homogeneous Maxwell equations. At the interface between the two dielectrics, the appropriate boundary conditions are applied to the general solutions of the fields which provide the necessary equations that are solved to obtain radiation fields. The energy radiated is further calculated using Poynting's theorem.

3. Calculation results

The solutions of the inhomogeneous Maxwell equations are found to be

$$\boldsymbol{E}_{1,2}(\boldsymbol{k}, \omega) = \frac{\mathrm{i}e\sigma_{1,2}^2}{8\pi^3 u_0 \varepsilon_{1,2}} \sum_{l=-\infty}^{+\infty} J_l(ak_z)[(\omega \boldsymbol{u}_0 n_{1,2}^2 - \boldsymbol{k})\delta(\chi_L - k_z) - \omega n_{1,2}^2 \boldsymbol{u}\Delta] \tag{7}$$

and

$$\boldsymbol{H}_{1,2}(\boldsymbol{k}, \omega) = (1/\omega\mu_{1,2})\boldsymbol{k} \times \boldsymbol{E}_{1,2} \tag{8}$$

where

$$\chi_L = \frac{\omega + l\omega_0}{u_0}, \qquad \chi_L^+ = \frac{\omega + (l+1)\omega_0}{u_0}, \qquad \chi_L^- = \frac{\omega + (l-1)\omega_0}{u_0},$$

$$\Delta = \tfrac{1}{2}[\delta(\chi_L^+ - k_z) + \delta(\chi_L^- - k_z)], \qquad \sigma_{1,2}^2 = (k^2 - \omega^2 n_{1,2}^2)^{-1},$$

and the suffix 1 or 2 refers to the medium under consideration. The solutions of the inhomogeneous Maxwell equations are written as

$$\boldsymbol{E}'_{1,2}(\boldsymbol{r}, t) = \int_{-\infty}^{+\infty} \boldsymbol{E}'_{1,2}(\boldsymbol{k}, \omega) \exp[\mathrm{i}(\chi\rho + \lambda_{1,2} z - \omega t)]\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega \tag{9}$$

and a similar expression for $\boldsymbol{H}'_{1,2}(\boldsymbol{r}, t)$. Here $\boldsymbol{\rho}$ and $\boldsymbol{\chi}$ are the components of vectors $\boldsymbol{r}$ and $\boldsymbol{k}$ in the xy plane, and λ is the z component of $\boldsymbol{k}$ satisfying the condition

$$\lambda_{1,2}^2 = \omega^2 n_{1,2}^2 - \chi^2. \tag{10}$$

Here λ is complex, i.e. $\lambda = \lambda' + \mathrm{i}\lambda''$. The radiation condition at infinity restricts the value of λ such that $\lambda_1' < 0$, $\lambda_1'' < 0$, $\lambda_2' > 0$ and $\lambda_2'' > 0$ because the first medium occupies the lower half space, while the second medium occupies the upper half space. The homogeneous Maxwell equations for the radiation fields yield

$$\boldsymbol{H}'_{1,2}(\boldsymbol{k}, \omega) = \frac{1}{\omega\mu_{1,2}}(\boldsymbol{\chi} + \hat{z}\lambda_{1,2}) \times \boldsymbol{E}'_{1,2}(\boldsymbol{k}, \omega)(\boldsymbol{\chi} + \hat{z}\lambda_{1,2})\boldsymbol{E}'_{1,2}(\boldsymbol{k}, \omega) = 0 \tag{11}$$

where $\hat{z}$ denotes the unit vector in the z direction. Resolving $\boldsymbol{E}'_{1,2}(\boldsymbol{k}, \omega)$ into its tangential and normal components, we rewrite the last condition as

$$\chi E'_{1,2t}(\boldsymbol{k}, \omega) + \lambda_{1,2} E'_{1,2n}(\boldsymbol{k}, \omega) = 0. \tag{12}$$

Using equations (7), (8), (11) and (12), the general solutions are obtained. Since the vectors $\boldsymbol{E}'_{1,2}(\boldsymbol{k}, \omega)$ are in the same direction as $\boldsymbol{\chi}$, only two of the four boundary conditions are independent. The solutions of the resulting two equations give the following radiation fields:

$$E'_{1t}(\boldsymbol{k},\omega)=(\chi\lambda_1/\zeta)G, \qquad E'_{1n}(\boldsymbol{k},\omega)=(-\chi^2/\zeta)G, \tag{13}$$

where

$$G=\frac{\mathrm{i}e}{8\pi^3}\sum_{l=-\infty}^{+\infty} J_l(ak_z)N_l(\chi),$$

$$N_l(\chi)=(1/u_0)[\eta(\chi)\delta(\chi_L-k_z)+a\omega_0\tilde{\eta}(\chi)\Delta],$$

$$\eta(\chi)=\left(\frac{\varepsilon_2}{\varepsilon_1}-\frac{\lambda_2}{\chi^2}\nu_1\right)\sigma_1^2+\left(-1+\frac{\lambda_2}{\chi^2}\nu_2\right)\sigma_2^2,$$

$$\tilde{\eta}(\chi)=(\omega\lambda_2/\chi^2)(n_1^2\sigma_1^2-n_2^2\sigma_2^2),$$

$$\nu_{1,2}=\omega n_{1,2}^2u_0-k_z,$$

$$\zeta=\varepsilon_2\lambda_1-\varepsilon_1\lambda_2.$$

The radiation fields in the second medium can be obtained if we interchange the indices 1 and 2 in equation (13). The expression for the radial component of the electric field, $E'_{1\rho}=E'_{1t}\cos\alpha$, where α is the angle between $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$, becomes

$$E'_{1\rho}=\frac{\mathrm{i}e}{8\pi^3}\int\frac{\chi\lambda_1\cos\alpha}{\zeta}\sum_{l=-\infty}^{+\infty}J_l(ak_z)N_l(\chi)$$
$$\times\exp[\mathrm{i}(\chi\rho\cos\alpha+\lambda_1 z-\omega t)]\chi\,\mathrm{d}\chi\,\mathrm{d}\alpha\,\mathrm{d}\omega\,\mathrm{d}k_z. \tag{14}$$

We integrate with respect to α from 0 *to* 2π, χ from 0 to ∞ and ω and k_z from $-\infty$ to $+\infty$.

The integral in α evaluated in terms of Bessel functions is

$$\int_0^{2\pi}\cos\alpha\exp(\mathrm{i}\chi\rho\cos\alpha)\,\mathrm{d}\alpha=2\pi\,\mathrm{i}J_1(\chi\rho).$$

Changing to spherical coordinates, using $\rho=R\sin\theta$, $z=-R\cos\theta$ (where R is the distance from the coordinate origin to the observation point), and for large values of R using the asymptotic form of the Bessel function, equation (14) can be put in the following form:

$$E'_{1\rho}=\frac{-e}{2\pi^2(2\pi R\sin\theta)^{1/2}}\sum_{l=-\infty}^{+\infty}\int\mathrm{d}k_z\,\mathrm{d}\omega\,J_l(ak_z)\,\mathrm{e}^{-\mathrm{i}\omega t}$$
$$\times\int\mathrm{d}\chi\frac{\chi^{3/2}\lambda_1}{\zeta}\{\exp[f(\chi)R-\tfrac{3}{4}\pi\mathrm{i}]+\exp[\phi(\chi)R+\tfrac{3}{4}\pi\mathrm{i}]\}N_l(\chi) \tag{15}$$

where

$$f(\chi)=\chi\sin\theta-\lambda_1R\cos\theta, \qquad \phi(\chi)=-\chi\sin\theta-\lambda_1R\cos\theta.$$

The integral with respect to χ in equation (15) can be evaluated by using the method of steepest descent (Garibyan 1958). Here we omit the details of the calculation and given the final result as

$$E'_{1\rho}=\frac{e}{2\pi^2R}\sum_{l=-\infty}^{+\infty}\int\mathrm{d}k_z\,\mathrm{d}\omega\,n_1^3\omega^2\sin\theta\cos^2\theta\exp[\mathrm{i}\omega(Rn_1-t)]$$
$$\times J_l(ak_z)[\mathscr{P}(k_z)\delta(\chi_L-k_z)+a\omega_0\tilde{\mathscr{P}}(k_z)\Delta] \tag{16}$$

where

$$\mathscr{P}(k_z)=[u_0(\varepsilon_1 M+\varepsilon_2 n_1\cos\theta)]^{-1}\left(\frac{\varepsilon_2/\varepsilon_1-(u_0M/\sin^2\theta)(n_1^2-k_z/\omega u_0)}{k_z^2-\omega^2 n_1^2\cos^2\theta}\right.$$
$$\left.+\frac{-1+(u_0M/\sin^2\theta)(n_2^2-k_z/\omega u_0)}{k_z^2-\omega^2M^2}\right),$$
$$\tilde{\mathscr{P}}(k_z)=\frac{M}{u_0 n_1^2\sin^2\theta}\left(\frac{n_1^2}{k_z^2-\omega^2 n_1^2\cos^2\theta}-\frac{n_2^2}{k_z^2-\omega^2M^2}\right),$$
$$M^2=n_2^2-n_1^2\sin^2\theta.$$

Applying a similar method of calculation to the normal component, we find that E'_{1n} satisfies a similar equation to that for E'_{1p} with $\sin^2\theta\cos\theta$ replacing $\sin\theta\cos^2\theta$. Using

$$E'_1=E'_{1p}\cos\theta+E'_{1n}\sin\theta,$$

and performing the integration with respect to k_z using the δ function, the radiation field in the first medium is found to be

$$E'_1=\frac{e}{2\pi^2Ru_0}\sum_{l=-\infty}^{+\infty}\int \mathrm{d}\omega n_1^3\omega^2\sin\theta\cos\theta\exp[\mathrm{i}\omega(Rn_1-t)]$$
$$\times\{J_l(a\chi_L)\mathscr{P}(\chi_L)+a\omega_0[J_l(a\chi_L^+)\mathscr{P}(\chi_L^+)+J_l(a\chi_L^-)\mathscr{P}(\chi_L^-)]\}. \tag{17}$$

The energy radiated by the charged particle in the solid angle $\mathrm{d}\Omega=\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ is given by

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega}=R^2\int_{-\infty}^{+\infty}E'_1H'^*_{1\phi}\,\mathrm{d}t=R^2\left(\frac{\varepsilon_1}{\mu_1}\right)^{1/2}\int_{-\infty}^{+\infty}E'_1E'^*_1\,\mathrm{d}t. \tag{18}$$

Using equations (17) and (18), assuming for simplicity the medium to be dispersionless, integrating with respect to t, ω and ω' with the help of the δ function, retaining only the terms linear in u, evaluating the infinite sums containing Bessel functions (Hansen 1975) and doing the somewhat tedious calculations of simplifying the various terms involved in equation (18), the final result can be put in the following form:

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega\,\mathrm{d}\omega}=\frac{e^2\varepsilon_1^{3/2}\mu_1^{5/2}\sin^2\theta\cos^2\theta}{4\pi^4}\frac{u_0^2(\varepsilon_2-\varepsilon_1)^2}{(\varepsilon_1M+\varepsilon_2n_1\cos\theta)^2}$$
$$\times\left|\frac{1-u_0^2[n_1^2\cos^2\theta+\varepsilon_2(n_2^2-n_1^2)/(\varepsilon_2-\varepsilon_1)]+u_0^3\varepsilon_1(n_2^2-n_1^2)M/(\varepsilon_2-\varepsilon_1)}{(1-u_0^2n_1^2\cos^2\theta)(1-u_0^2M^2)(1-\alpha/\gamma^2)}\right|^2. \tag{19}$$

The above result specifies the intensity of backward transition radiation (i.e. the radiation going in the medium −1) at an angle θ wrt z per unit solid angle per unit frequency interval in the presence of the alternating electric field due to a relativistically moving charged particle crossing the boundary between two media. Under the same physical situation, forward transition radiation (i.e. the radiation going in the medium −2) can be obtained from equation (19) by interchanging the indices 1 and 2, and replacing u_0 by $-u_0$ (as now $-z$ can be conveniently taken as the reference axis with no other change). Here we conclude that the backward and the forward transition radiations are not equal.

The final result, namely equation (20) of DT, taking almost three-quarters of a page in length, is made up of six complicated terms, each consisting of infinite sums and integrals of expressions which in their turn involve six lengthy functions. Because of a wrong factor, (ω^2/c^3), in the second terms of equations (15) and (17) of DT, the velocity and the frequency dependences in all the remaining five terms, except the first, are found to be incorrect. Further, while analysing their equation (20), the authors have used inconsistent approximations, i.e. they have neglected terms $\sim\beta^2$ and higher and simultaneously have retained the terms $\sim(u^2/u_0^2)$ which are of lower order even for non-relativistic velocities. Consequently, in all the special cases, namely equations (22), (23), (24) and (25), of equation (20) of DT, the dominant linear field-dependent terms have been omitted by them and they have arrived at wrong conclusions.

In the absence of the field (i.e. $u = 0$), for the case of any two non-magnetic media ($\mu_1 = \mu_2 = 1$), our result given by equation (19) reduces to the relativistic expression, namely equation (20) of Zrelov and Ruzicka (1978) and equations (2.35) and (2.37) of Ginzburg and Tsytovich (1979).

In the case of a particle entering a dielectric from a vaccum, the zero-field case of our result for backward transition radiation after substitutions $\varepsilon_1 = 1 = \mu_1 = \mu_2$, $\varepsilon_2 = \varepsilon$ reduces to equation (21) of Garibyan (1958). For a particle entering a vacuum from a dielectric, the zero-field case of our result for forward transition radiation after substitutions $\varepsilon_2 = 1 = \mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon$ matches with equation (29) of Garibyan (1958). For non-relativistic velocities, our results, as above, coincide with equation (37) of Frank and Ginzburg (1945).

Equation (19) is the most general expression that specifies the intensity of transition radiation in the presence of an alternating electric field which is valid for any velocity of the particle crossing the boundary between any two media, as long as the modulation energy is very small compared with the mean particle energy (i.e. $u' \ll u_0$).

4. Discussion and conclusion

We can put equation (19) in short as

$$\mathrm{d}w/\mathrm{d}\Omega\,\mathrm{d}\omega = I = I_0(1-\alpha/\gamma^2)^{-2} \simeq I_0(1+2\alpha/\gamma^2) \tag{20}$$

where I_0 denotes the intensity of the backward transition radiation without a field and the binomial expansion is valid since $\alpha \ll 1$, $\gamma \geqslant 1$. Equation (20) shows that in the presence of the field, the backward transition radiation increases, but due to the change in sign of u_0 the forward transition radiation decreases.

Writing the field parameters explicitly in equation (20), we obtain

$$I = I_0\left[1+\frac{2e}{m_0}\left(\frac{E_0}{\omega_0}\right)\left(\frac{1}{u_0\gamma^3}\right)\right]. \tag{20$'$}$$

We can therefore conclude that the field contribution to the radiation varies linearly with the field strength and that it varies inversely with the field frequency. The conclusion drawn by DT in this context regarding a slight reduction in high-frequency fields and quadratic increase due to weak low-frequency fields is wrong. In the case of non-relativistic velocities ($\gamma = 1$) the field contribution varies inversely with the particle velocity. For relativistic ($\gamma > 1$) and ultra-relativistic ($\gamma \gg 1$) velocities, owing to the factor γ^{-3} in equation (20′), the field contribution to the radiation is comparatively small.

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